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Computational methods for some non-linear wave equations

Q. CAO[1,2], K. DJIDJELI[2], W. G. PRICE[2] and E. H. TWIZELL[3*]
[1]*Department of Mathematics and Physics, Shandong University of Technology, Jinan, China, 250014*
[2]*Department of Ship Science, University of Southampton, Southampton UK, SO17 1BJ*
[3]*Department of Mathematics and Statistics, Brunel University, Uxbridge, Middlesex UK, UB8 3PH*

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Abstract. Computational methods based on a linearized implicit scheme and a predictor-corrector method are proposed for the solution of the Kadomtsev–Petviashvili (KP) equation and its generalized from (GKP). The methods developed for the KP equation are applied with minor modifications to the generalized case.

An important advantage to be gained from the use of the linearized implicit method over the predictor-corrector method which is conditionally stable, is the ability to vary the mesh length, and thereby reducing the computational time.

The methods are analysed with respect to stability criteria. Numerical results portraying a single line-soliton solution and the interaction of two-line solitons are reported for the KP equation. Moreover, a lump-like soliton (a solitary wave which decays to zero in all space dimensions) and the interaction of two lump solitons are reported for the KP equation.

Key words: computational methods, KP/GKP equation, solitary waves, stability analysis, phase error.

1. Introduction

Although Scott–Russell [1] observed solitary waves (or solitons) travelling on the calm water surface of a canal more than one hundred and fifty years ago, it is only in the past thirty years that interest has focussed on the existence, properties and the dynamical interactive behaviour of solitons [2]. This has arisen because many physical phenomena of non-linear dynamical systems can be described by a soliton model [2, 3] such as, for example, the Korteweg-de Vries equation (KdV) [4], the two-dimensional Korteweg-de Vries equation or the Kadomtsev–Petviashvili equation [5], the sine-Gordon (SG) model and the non-linear Schrödinger equation [2]. In fact, the third-order KdV equation has been shown to model one-dimensional waves of small but finite amplitude in dispersive systems, for example, shallow-water waves [4] and free surface flow over an obstacle [6]. The KP equation models two-dimensional shallow-water waves in confined [7, 8] or open water [9]. In the last few years, a considerable amount of research activity has focussed on the generation of non-linear waves by moving sources. The experimental observations of Huang *et al.* [10] provided the original motivation, whereby they point out that ships travelling in channels of finite depth and width at near-critical speeds, that is, close to the linear-long-wave speed, continuously excite solitons which form and propagate upstream. Theoretical evidence for the excitation of solitons near critical conditions was first provided by the numerical calculations of Wu and Wu [11], who used Boussinesq equations to study a pressure patch moving on the free surface at a near-critical speed and found upstream solitons emitted periodically. Since then various

* To whom all correspondence and requests for off-prints should be addressed.

investigations on the description of two- and three-dimensional non-linear water waves have followed. Using the basic equations of fluid dynamics (incompressible Navier–Stokes equations), it has been shown that the generation of three-dimensional non-linear water waves by moving sources can be described by the forced KP equation (see, for instance, Katsis and Akylas [12] for free surface flow over an obstacle, and Choi and Mei [13] for free surface waves generated by a moving slender ship).

Through developments in operator theory and numerical analysis, it has been demonstrated that a class of non-linear differential equations admits solutions in the form of solitons. For example, analytical approaches based on the inverse scattering method [14], the Lax operator method [15] and the method of Hirota [16] have permitted the evaluation of solutions to physical dynamical systems modelled by non-linear differential equations (see, for example, Drazin [2]). Through these various means, descriptions of the dynamical behaviour of many complex physical systems exhibiting soliton-type solutions have been achieved. The application, however, of these methods to a specific problem is not straightforward as only a restricted number of special cases can be solved using the available techniques. Moreover, there are many examples of inexact, or quasi-soliton behaviour where little or no analytical results are known and thus numerical studies are essential in order to develop an understanding of the phenomena.

The numerical analysis literature on numerical methods for multi-dimensional soliton equations is sparse compared to that of one-dimensional soliton equations. For example, for the KdV equations, several numerical schemes have been proposed and used successfully in recent years (see, for instance, Fornberg and Whitham [17], Taha and Ablowitz [18], Djidjeli *et al.*, [19] and references therein) but the numerical solution of KP does not seem to have been so successful [12]. One reason for this might be due to the fact that the conditions for the stability of the explicit schemes (when used to solve the KP equation) are very restrictive (see, for instance, the predictor-corrector method in Section 2). Katsis and Akylas [12] presented an explicit method based on the Lax–Wendroff scheme; more recently, Chen and Sharma [20] used in their work a finite-difference method based on a fractional step algorithm with Crank–Nicolson schemes in each half step, while Wang *et al.* [21] and Minzoni and Smyth [22] used the pseudo-spectral method of Fornberg and Whitham [17].

In the present paper a numerical method based on the linearized implicit scheme is presented for the generalized KdV (GKdV) equation and then by a straightforward extension this method is applied to the KP and GKP equations. Moreover, an alternative method based on the predictor-corrector scheme (where the implicit scheme is used as a corrector part of the predictor-corrector method) is also presented for the KP/GKP equation. The numerical methods are analysed with respect to linear stability theory. It is shown that the linearized implicit scheme is unconditionally stable, while the predictor-corrector is conditionally stable. Numerical results for a line/lump soliton and two line/lump solitons for the KP equations are presented in Section 3.

2. Computational methods for the KP equation and its generalized form

The KP equation and its generalized form (GKP) are given by

$$(u_t + \beta u u_x + \mu u_{xxx})_x - \delta u_{yy} = 0, \qquad (u_t + \beta u^\alpha u_x + \mu u_{xxx})_x - \delta u_{yy} = 0, \tag{1}$$

respectively, where β, μ and δ are constants. These equations are of considerable interest particularly for $\alpha = 1$ (KP equation) and $\alpha = 2$, because they arise in a large number of physical applications, for example long waves on the surface of a fluid [5], and long internal waves in a fluid [23].

For $\delta = 0$, equation (1) reduces to the KdV/GKdV equation [17]. It is helpful to present first the linearized implicit method for the GKdV, and then to extend it to the KP equation. The scheme which is developed for the KP equation is then applied with minor modifications to the GKP equation.

Rewriting the GKdV equation (Equation (1) with $\delta = 0$) as

$$u_t + \frac{\beta}{\alpha+1}(u^{\alpha+1})_x + \mu u_{xxx} = 0, \tag{2}$$

the linearized implicit method, which was developed previously [19] for the KdV equation, we will extend for the generalized KdV equation here.

The implicit formulation for solving Equation (2) is given by

$$\begin{aligned} U_m^{n+1} \;=\; & U_m^n - \frac{\beta\Delta t}{2(\alpha+1)}(F_m^n + F_m^{n+1})_x \\ & -\tfrac{1}{2}\mu\Delta t(U_m^n + U_m^{n+1})_{xxx}, \end{aligned} \tag{3}$$

where $F_m^n = (U_m^n)^{\alpha+1}$, $U_m^n \approx u(m\Delta x, n\Delta t)$ and Δx and Δt are space and time steps. The linearized form is obtained simply by using Taylor's expansion of F_m^{n+1} about the nth-time level, and is given by

$$aU_{m-2}^{n+1} + b_m^n U_{m-1}^{n+1} + U_m^{n+1} + c_m^n U_{m+1}^{n+1} - aU_{m+2}^{n+1} = d_m^n, \tag{4}$$

where

$$a = -\frac{\mu\Delta t}{4\Delta x^3}, \qquad b_m^n = -\tfrac{1}{4}\beta\frac{\Delta t}{\Delta x}(U_{m-1}^n)^\alpha + \frac{\mu\Delta t}{2\Delta x^3},$$

$$c_m^n = \tfrac{1}{4}\beta\frac{\Delta t}{\Delta x}(U_{m+1}^n)^\alpha - \frac{\mu\Delta t}{2\Delta x^3}$$

and

$$\begin{aligned} d_m^n \;=\; & U_m^n - \frac{\mu\Delta t}{4\Delta x^3}(U_{m+2}^n - 2U_{m+1}^n + 2U_{m-1}^n - U_{m-2}^n) \\ & - \frac{\beta(1-\alpha)}{4(1+\alpha)}\frac{\Delta t}{\Delta x}\left[(U_{m+1}^n)^{\alpha+1} - (U_{m-1}^n)^{\alpha+1}\right]. \end{aligned}$$

In obtaining Equation (4), we will replace the space derivatives in (2) by their central difference approximations.

Computationally, this implicit finite-difference method is economical in that it permits the numerical solution of the nonlinear differential equation (2) to be obtained by solving a single, linear, algebraic system with penta-diagonal matrix at each time step (using LU decomposition).

When the determinant of the matrix of the algebraic system is close to zero, an ill-posed problem arises. It is worth noting, however, that, for an algebraic system with penta-diagonal matrix, the main correctness condition of the numerical solution is the condition of the diagonal dominance of the matrix [24]. For the algebraic system (4), this is given by

$$\frac{\Delta t}{\Delta x^3}[\mu + |\tfrac{1}{2}\beta \Delta x^2 (U^n_{m-1})^\alpha - \mu| + |\tfrac{1}{2}\beta \Delta x^2 (U^n_{m+1})^\alpha - \mu|] < 2. \tag{5}$$

Violation of condition (5) implies the possibility of round-off error destroying the inversion matrix. However, the main point that should be noted is that condition (5) does not arise from a stability analysis, and therefore it is not a restriction on the implicit method (4). A different inversion might be necessary when condition (5) is violated.

In an attempt to gain some insight into the stability of the method, we use linearized stability to analyse the method. It is known that in the early stages, an instability develops in a very small region. Therefore, if the solution is slowly varying, an instability may be predicted by means of a stability analysis of a localized version of the difference scheme. By freezing temporarily the nonlinear term in Equation (3) and using the von Neumann method, we can show that the amplification factor $g(k_1)$ for the method (3) is given by

$$g(k_1) = \frac{1 - iC}{1 + iC}, \tag{6}$$

in which

$$C = \frac{\Delta t}{\Delta x} \sin \xi \left[\tfrac{1}{2}\beta\nu - \frac{\mu}{\Delta x^2}(1 - \cos \xi)\right],$$

$\nu = \max_m |(U_m(t_0))^\alpha|$ and $\xi = k_1 \Delta x$, where $k_1 = 2\pi/L$ is the wave number, L is the wavelength. Since $|g(k_1)| = 1$, method (3) is stable and non-dissipative. For wave simulations (particularly higher waves), it is important to analyse the phase error (dispersion) as large phase error can produce solutions that are totally out of phase with the (unknown) exact solution: that is, a meaningless solution would then be obtained. In any discretization procedure only long waves can be approximated well. Thus, the phase error of the higher-frequency components is of little significance, and the main interest is in sufficiently small ξ.

The numerical phase of the method (3) is given by

$$P_N(\xi) = \tan^{-1}\left[\mathrm{Im}\frac{\{g(k_1)\}}{\mathrm{Re}\{g(k_1)\}}\right] = -\tan^{-1}\left\{\frac{2C}{1 - C^2}\right\}. \tag{7}$$

The phase error is given by

$$\begin{aligned} E(\xi) &= P_N(\xi) + \Delta t.l(k_1) \\ &= -\tan^{-1}\left\{\frac{2A}{1 - A^2}\right\} + \frac{\Delta t}{\Delta x}\xi\left[\beta\nu - \frac{\mu}{\Delta x^2}\xi^2\right], \end{aligned} \tag{8}$$

where $l(k_1)$ is the dispersion relation for Equation (2) with constant coefficients. For small ξ, it can be shown using Taylor's series that the phase error (8) reduces to

$$\begin{aligned} E(\xi) &= s\xi\Bigg[(\tfrac{1}{6}\beta\nu + \tfrac{1}{12}\beta^3 s^2 \nu^3)\xi^2 + \Bigg(\tfrac{3}{16}s^4\beta^5\nu^5 - \tfrac{1}{24}s^2\beta^4\nu^3 \\ &\qquad - \frac{1}{4\Delta x^2}s^2\beta^2\nu^2\mu - \frac{1}{6\Delta x^2}\mu\Bigg)\xi^4\Bigg] + O(\xi^6), \quad \text{as } \xi \to 0, \end{aligned} \tag{9}$$

where $s = \Delta t/\Delta x$.

The corresponding phase speed error is given by

$$\begin{aligned} E_v(k_1) &= \frac{l(k_1)}{k_1} - \frac{\Delta x}{\xi \Delta t} \tan^{-1}\left\{\frac{2C}{1-C^2}\right\} \\ &= \beta\nu - (\mu + \tfrac{1}{6}\beta\nu\Delta x^2 + \tfrac{1}{12}\beta^3\Delta t^2\nu^3)k_1^2 + O(k_1^4) \quad \text{as } k_1 \to 0. \end{aligned} \tag{10}$$

Modes with low wavenumbers correspond well to their counterparts in the analytic solution. The modes with higher wavenumbers (which usually arise when there is a discontinuity in the coefficients of the differential equation or in the initial data) will show a marked difference. A way of eliminating the effect of the unvoidable high wavenumber components is to add an additional artificial dissipation to the scheme without affecting the order of accuracy. These types of problem occur in non-linear hyperbolic systems in which shocks occur and dissipation is required to produce a narrow wave front [25].

Turning next to the KP equation, by rewriting the KP Equation (1) as

$$u_{tx} + \frac{\beta}{2}(u^2)_{xx} + \mu u_{xxxx} - \delta u_{yy} = 0$$

and using an implicit formulation as in the GKdV equation, we find that

$$\begin{aligned} &U_{m,k}^{n+1} - U_{m-1,k}^{n+1} \\ &\quad = U_{m,k}^n - U_{m-1,k}^n - \tfrac{1}{2}\beta\Delta t\Delta x[(U_{m,k}^n)^2 + (U_{m,k}^{n+1})^2]_{xx} \\ &\qquad -\mu\Delta t\Delta x(U_{m,k}^n + U_{m,k}^{n+1})_{xxxx} + \delta\Delta t\Delta x(U_{m,k}^n + U_{m,k}^{n+1})_{yy}. \end{aligned} \tag{11}$$

Using Taylor's expansion as for the GKdV equation, we observe that the linearized implicit method for solving the KP Equation (1) becomes

$$\begin{aligned} &pU_{m+2,k}^{n+1} + r_{m,k}^n U_{m+1,k}^{n+1} + s_{mk}^n U_{m,k}^{n+1} + t_{mk}^n U_{m-1,k}^{n+1} \\ &\quad + pU_{m-2,k}^{n+1} - qU_{m,k+1}^{n+1} - qU_{m,k-1}^{n+1} = b_{mk}^n, \end{aligned} \tag{12}$$

where

$$\begin{aligned} b_{mk}^n &= -pU_{m+2,k}^n + (1+4p)U_{m+1,k}^n + (-6p-2q)U_{m,k}^n \\ &\quad + (-1+4p)U_{m-1,k}^n - pU_{m-2,k}^n + qU_{m,k+1}^n + qU_{m,k-1}^n, \end{aligned}$$

and

$$r_{mk}^n = 1 - 4p + \beta r U_{m+1,k}^n, \qquad s_{mk}^n = 6p + 2q - 2\beta r U_{m,k}^n,$$

$$t_{mk}^n = -1 - 4p + \beta r U_{m-1,k}^n, \qquad r = \frac{\Delta t}{\Delta x}, \quad p = \mu\frac{\Delta t}{\Delta x^3}, \quad q = \delta\frac{\Delta t\Delta x}{\Delta y^2}.$$

In a similar way, the numerical scheme developed for the KP equation can be applied with minor modification to the generalized case. That is, the numerical method for solving the GKP equation is now given by Equation (12) with

$$\begin{aligned} b_{mk}^{n} \;=\; & -pU_{m+2,k}^{n} + \left[1 + 4p - \frac{\beta(1-\alpha)r}{(1+\alpha)}(U_{m+1,k}^{n})^{\alpha}\right]U_{m+1,k}^{n} \\ & + \left[-6p - 2q + \frac{2\beta(1-\alpha)r}{(1+\alpha)}(U_{m,k}^{n})^{\alpha}\right]U_{m,k}^{n} \\ & + \left[-1 + 4p - \frac{\beta(1-\alpha)r}{(1+\alpha)}(U_{m-1,k}^{n})^{\alpha}\right]U_{m-1,k}^{n} \\ & - pU_{m-2,k}^{n} + qU_{m,k+1}^{n} + qU_{m,k-1}^{n}, \end{aligned}$$

and

$$r_{mk}^{n} = 1 - 4p + \beta r(U_{m+1,k}^{n})^{\alpha}, \qquad s_{mk}^{n} = 6p + 2q - 2\beta r(U_{m,k}^{n})^{\alpha},$$

$$t_{mk}^{n} = -1 - 4p + \beta r(U_{m-1,k}^{n})^{\alpha}.$$

Using the linearized stability analysis, it can be shown, as for the GKdV equation, that method (11) is unconditionally stable. Solving the sparse linear system given by Equation (12) (for the case of a line/lump soliton considered in Section 3), using iterative methods such as the Jacobi method or the Successive Overrelaxation method (SOR), or the ITPACK software package of subroutines [26], we obtain solutions that remain bounded for a period of time before experiencing a sudden growth leading to overflow within a few time steps. This may be due to the fact that the matrix given by the algebraic system (12) is not diagonally dominant. To overcome this problem, an iteration scheme of the form

$$\begin{aligned} & p(U_{m+2,k}^{n+1})^{[l]} + r_{m,k}^{n}(U_{m+1,k}^{n+1})^{[l]} + s_{mk}^{n}(U_{m,k}^{n+1})^{[l]} \\ & \quad + t_{mk}^{n}(U_{m-1,k}^{n+1})^{[l]} p(U_{m-2,k}^{n+1})^{[l]} \\ & \quad = b_{mk}^{n} + q(U_{m,k+1}^{n+1})^{[l-1]} + q(U_{m,k-1}^{n+1})^{[l-1]}, \quad l = 1, 2, \ldots \end{aligned} \tag{13}$$

is used to solve the linear system (12), with an initial approximation $U^{[0]}$ at each time step taken to be $(U_{m,k}^{n+1})^{[0]} = U_{m,k}^{n}$. Equation (13) corresponds to a linear system with penta-diagonal matrix (as in the GKdV equation), and is solved by LU decomposition. Computationally, this approach is found to be economical, since only a few iterations are needed at each time step for the method to attain convergence. The criterion used for stopping the iterations at each time step is

$$\frac{\|\mathbf{U}^{[l]} - \mathbf{U}^{[l-1]}\|}{\|\mathbf{U}^{[l-1]}\|} < \varepsilon.$$

The algorithm (13) is suited to parallel computation. At each iteration, the linear system (13) of order $N(m = 1, 2, \ldots, N)$ for $k = 1, 2, \ldots, L$ can be solved simultaneously using different processors (as each system of order N for a given value of $k(k = 1, \ldots, L)$ is independent of the others).

An alternative approach for solving Equation (1) is to use the non-linear implicit scheme (11) as a corrector part of a perdictor–corrector method. This method (the predictor–corrector) is explicit (and hence, there is no algebraic system to be solved at each time step), but this unfortunately is at the expense of a stability condition which places limits on the discretization parameters.

The predictor–corrector method is given by

$$\begin{aligned}\text{Predictor: } & \bar{U}^{n+1}_{m+1,k} - \bar{U}^{n+1}_{m-1,k} \\ & = U^n_{m+1,k} - U^n_{m-1,k} + 2\Delta t[(1-\eta) f(U^{n-1}_{m,k}) + \eta f(U^n_{m,k})],\end{aligned} \tag{14}$$

$$\begin{aligned}\text{Corrector: } & U^{n+1}_{m+1,k} - U^{n+1}_{m-1,k} \\ & = U^n_{m+1,k} - U^n_{m-1,k} + \Delta t[f(U^n_{m,k}) + f(\bar{U}^{n+1}_{m,k})],\end{aligned} \tag{15}$$

where

$$\begin{aligned} f(U^n_{m,k}) &= -\frac{\mu}{\Delta x^3} U^n_{m+2,k} + \left[\frac{4\mu}{\Delta x^3} - \frac{\beta}{(\alpha+1)\Delta x}(U^n_{m+1,k})^\alpha\right] U^n_{m+1,k} \\ &\quad - \left[\frac{6\mu}{\Delta x^3} + \frac{2\delta\Delta x}{\Delta y^2} - \frac{2\beta}{(\alpha+1)\Delta x}(U^n_{m,k})^\alpha\right] U^n_{m,k} \\ &\quad + \left[\frac{4\mu}{\Delta x^3} - \frac{\beta}{(\alpha+1)\Delta x}(U^n_{m-1,k})^\alpha\right] U^n_{m-1,k} \\ &\quad - \frac{\mu}{\Delta x^3} U^n_{m-2,k} + \delta \frac{\Delta x}{\Delta y^2}(U^n_{m,k-1} + U^n_{m,k+1}). \end{aligned}$$

As before, to gain an insight into the stability of the method (14)–(15), we use linear stability theory. By combining the predictor–corrector algorithm given by Equations (14), (15), we may show after some manipulation that

$$\begin{aligned} \mathbf{U}^{n+1} &= (I + 2\Delta t \Delta x M + 2\eta \Delta t^2 \Delta x^2 M^2)\mathbf{U}^n \\ &\quad + 2(1-\eta)\Delta t^2 \Delta x^2 M^2 \mathbf{U}^{n-1}, \end{aligned} \tag{16}$$

where $\mathbf{U}(t) = [U_{11}(t), U_{21}(t), \ldots, U_{N1}(t); \ldots; U_{1N}(t), U_{2N}(t), \ldots, U_{NL}(t)]^T$, T denoting transpose, is a vector of order NL (with $N = (X_1 - X_0)/\Delta x - 1$, $L = (Y_1 - Y_0)/\Delta y - 1$) and M is a matrix of order NL given by

$$M = \begin{bmatrix} B & \frac{\delta}{\Delta y^2}D^{-1} & & & \\ \frac{\delta}{\Delta y^2}D^{-1} & B & \frac{\delta}{\Delta y^2}D^{-1} & & \\ & & \cdot & \cdot & \\ & & \frac{\delta}{\Delta y^2}D^{-1} & B & \frac{\delta}{\Delta y^2}D^{-1} \\ & & & \frac{\delta}{\Delta y^2}D^{-1} & B \end{bmatrix}.$$

Here B is a matrix of order N and is given by

$$B = D^{-1}\left[-\frac{\mu}{\Delta x^4}Q^2 - \frac{\beta\nu}{(\alpha+1)\Delta x^2}Q - \frac{2\delta}{\Delta y^2}I\right],$$

with $\nu = \max|(U_{m,k}(t_0))^\alpha|$, I is the identity matrix of order N, and D and Q are matrices of order N given by

$$D = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & . & . & . & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & . & . & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}.$$

The order N of the matrix D should be even so that the matrix D^{-1} exists.

In obtaining the matrices M, D and Q, we use the boundary conditions $u(x, y, t)$ and $\partial^2 u/\partial x^2(x, y, t)$ at $x = X_0$ and $x = X_1$, and $u(x, y, t)$ at $y = Y_0$ and $y = Y_1$ (for certain types of problems, *e.g.* the lump soliton solutions cosidered in Section 3, the above boundary conditions can be set to zero). For other problems requiring other types of boundary conditions, the matrices M, D and Q may be derived by considering method (14–15) only for the interior points. This assumes that the exclusion of these types of boundary conditions do not have a significant influence on the overall stability limit which may be the case at least for moderate solution gradients near the boundaries. It is worth noting that the matrix which arises from the approximation of the space derivative $\partial^4 u/\partial x^4$ by its central-difference approximant is simply the square of the matrix Q obtained from the approximation of the space derivative $\partial^2 u/\partial x^2$ by its central-difference approximant. This can be seen simply by replacing $u(X_0 - \Delta x, y, t)$ and $u(X_1 + \Delta x, y, t)$ (which arise from the space derivative $\partial^4 u/\partial x^4$ replaced by its central-difference approximant at $m = 1$ and $m = N$) by

$$\begin{aligned} u(X_0 - \Delta x, y, t) &= -u(X_0 + \Delta x, y, t) + 2u(X_0, y, t) \\ &\quad + \Delta x^2 \frac{\partial^2 u(X_0, y, t)}{\partial x^2} \end{aligned}$$

and

$$\begin{aligned} u(X_1 + \Delta x, y, t) &= -u(X_1 - \Delta x, y, t) + 2u(X_1, y, t) \\ &\quad + \Delta x^2 \frac{\partial^2 u(X_1, y, t)}{\partial x^2} \end{aligned}$$

Equation (16) may be written as

$$\mathbf{Z}^{n+1} = W\mathbf{Z}^n, \tag{17}$$

where $\mathbf{Z}^{n+1} = [(\mathbf{U}^n)^T, (\mathbf{U}^{n+1})^T]^T$, W is the matrix

$$W = \begin{bmatrix} O & I \\ S & R \end{bmatrix}$$

and I is the identity matrix of order NL. The matrices S and R are of order NL and are given by

$$S = 2(1-\eta)\Delta t^2 \Delta x^2 M^2 \quad \text{and}$$

$$R = I + 2\Delta t \Delta x M + 2\eta \Delta t^2 \Delta x^2 M^2.$$

The global error in (17) will not grow as $n \to \infty$ if the eigenvalues of the matrix W are less than unity in modulus. The eigenvalues of the matrix W are given by the solution of the equation

$$\phi(\lambda) = \lambda^2 - \lambda_R \lambda - \lambda_S = 0, \tag{18}$$

where λ_R and λ_S are the eigenvalues of matrices R and S and are given by

$$\lambda_R = 1 + 2\Delta t \Delta x \lambda_M + 2\eta \Delta t^2 \Delta x^2 \lambda_M^2 \quad \text{and}$$

$$\lambda_S = 2(1-\eta)\Delta t^2 \Delta x^2 \lambda_M^2,$$

with λ_M an eigenvalue of the matrix M.

It can be shown that the NL eigenvalues of M are given by

$$\lambda_{j,l} = i\zeta_{j,l}, \quad i = \sqrt{-1}, \tag{19}$$

where

$$\zeta_{j,l} = \frac{2}{\cos \frac{j\pi}{N+1}} \left[\frac{4\mu}{\Delta x^4} \sin^4 \frac{j\pi}{2(N+1)} - \frac{\beta\nu}{(\alpha+1)\Delta x^2} \sin^2 \frac{j\pi}{2(N+1)} + \frac{\delta}{\Delta y^2} \sin^2 \frac{l\pi}{2(N+1)} \right], \quad j = 1, 2, \ldots, N; \ \ l = 1, 2, \ldots, L.$$

Using the Schur criterion (see, for instance, Lambert [27]), we find that the roots of Equation (18) satisfy $|\lambda_r| < 1$, $r = 1, 2$ if

$$|\hat{\phi}(0)| > |\phi(0)|, \tag{20}$$

where

$$\hat{\phi}(\lambda) = -\lambda_S \lambda^2 - (\lambda_R)^* \lambda + 1,$$

and the linear polynomial

$$\phi_1(\lambda) = \frac{1}{\lambda}[\hat{\phi}(0)\phi(\lambda) - \phi(0)\hat{\phi}(\lambda)] = (1 - \lambda_S^2)\lambda - \lambda_R - (\lambda_R)^* \lambda_S, \tag{21}$$

is a Schur polynomial (that is, its root is less than unity in modulus). In Equations (20–21), $(\lambda_R)^*$ is the complex conjugate of λ_R.

Using inequality (20), we have

$$8|1-\eta| \left[4\mu \frac{\Delta t}{\Delta x^3} \sin^4 \frac{j\pi}{2(N+1)} - \frac{\beta\nu}{(\alpha+1)} \frac{\Delta t}{\Delta x} \sin^2 \frac{j\pi}{2(N+1)} + \frac{\delta \Delta t \Delta x}{\Delta y^2} \sin^2 \frac{l\pi}{2(N+1)} \right]^2 + \sin^2 \frac{j\pi}{(N+1)} < 1. \tag{22}$$

Using now Equation (21), we observe that $\phi_1(\lambda)$ is a Schur polynomial if

$$\left| \lambda = \frac{\lambda_R + (\lambda_R)^* \lambda_S}{1 - \lambda_S^2} \right| < 1. \tag{23}$$

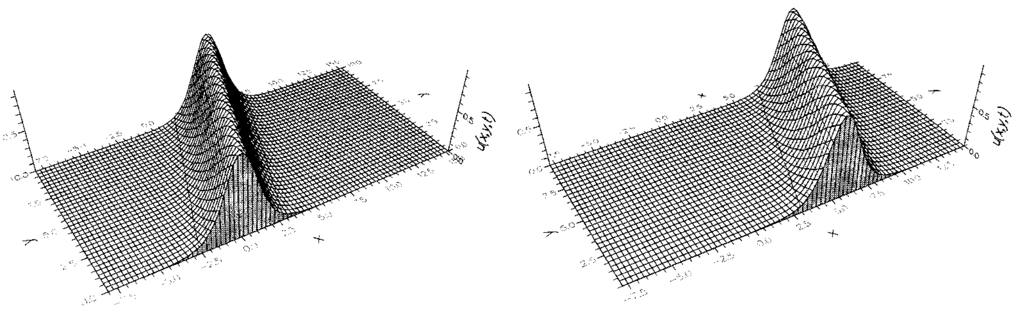

Figure 1. The initial condition (25) and the solitary wave at time $t = 3$ for the KP equation.

If we replace λ_R, $(\lambda_R)^*$ and λ_S by their values in (23), then

$$4\eta^2(1-\eta)^2(2\eta-1)\rho_{j,l}^4 + 4(1-\eta)(2\eta-1)\rho_{j,l}^2 + 7 - 6\eta < 0, \tag{24}$$

where

$$\rho_{j,l} = \Delta t \Delta x \zeta_{j,k} = \frac{2}{\cos \frac{j\pi}{N+1}} \left[4\mu \frac{\Delta t}{\Delta x^3} \sin^4 \frac{j\pi}{2(N+1)} - \frac{\beta \nu}{(\alpha+1)} \frac{\Delta t}{\Delta x} \sin^2 \frac{j\pi}{2(N+1)} + \frac{\delta \Delta t \Delta x}{\Delta y^2} \sin^2 \frac{l\pi}{2(N+1)} \right]$$

3. Computational results

Numerical results describing a single soliton solution and the interaction of two solitons for the KP equation (GKP equation with $\alpha = 1$) are presented. The numerical results obtained are compared to the analytical solution (whenever the analytical solution is known), each providing corroborative evidence of the findings of the other.

KP one and two line-soliton solutions

Figure 1 illustrates the solitary waves for $\beta = 6$, $\mu = 1$, $\delta = 1$, $k_x = 0{\cdot}7$, $k_y = -0{\cdot}3$, $\Delta t = 0{\cdot}05$, $\Delta x = 0{\cdot}2$, $\Delta y = 0{\cdot}5$ and tolerance $\varepsilon = 0{\cdot}0001$ at time $t = 3$ using the linearized implicit scheme (13), together with an initial condition given by

$$u(x, y, t = 0) = \{A \operatorname{sech}^2(k_x x + k_y y - \omega t - x_0)\}^{1/\alpha}, \tag{25}$$

where

$$A = \frac{2(\alpha+1)(\alpha+2)}{\beta\alpha^2}\mu k_x^2, \qquad \omega = \frac{4\mu k_x^3}{\alpha^2} - \delta(k_y^2/k_x).$$

and x_0 is constant taken to be zero here.

The results obtained are found to be in a good agreement with the analytical solution. The CPU time needed to reach $t = 3$ on a Sun SPARC Station 10 was 44·23s.

For the treatment of numerical boundary conditions, the boundary conditions along the y-axis ($k = 1, 2, \ldots, L$) were taken as

$$U_{m,k}^n = U_{m,k}^{n+1} = 0 \quad \text{for } m = -1, 0, 1 \quad \text{and} \quad m = N, N+1, N+2. \tag{26}$$

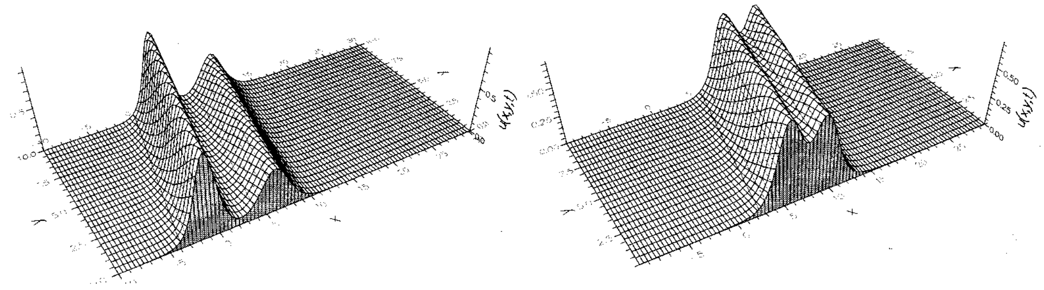

Figure 2. The initial condition and the two solitary wave interaction at time $t = 4$ for the KP equation.

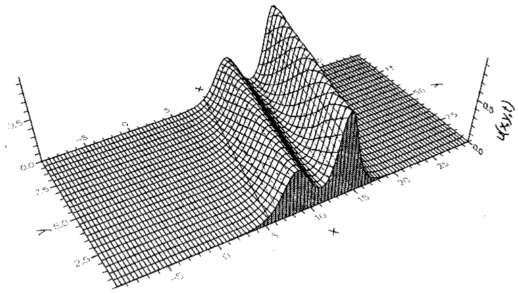

Figure 3. Two solitary wave interaction at time $t = 7$ for the KP equation.

Along the x-axis, however, the treatment of the numerical boundary conditions for the line soliton is not straightforward as the solution does not vanish in all directions. In the present study, exact boundary conditions are used on the x-axis with the aim of testing the effectiveness of the proposed numerical methods. In future work, focus will be placed on trying to find the numerical boundary conditions using for example, an appropriate interpolation/extrapolation technique.

Figures 2–3 show the interaction of two line-soliton solutions at times $t = 0$ (initial conditions [28]), $t = 4$ and $t = 7$, respectively, for $k_{x_1} = 0{\cdot}7$, $k_{x_2} = 0{\cdot}5$, $k_{y_1} = -0{\cdot}3$, $k_{y_2} = -0{\cdot}2$, $x_1 = 0$, $x_2 = 2$, $\Delta t = 0{\cdot}05$, $\Delta x = 0{\cdot}2$, $\Delta y = 0{\cdot}5$ and $\varepsilon = 0{\cdot}0001$, using the linearized implicit scheme (13). From these figures, it can be seen that the faster pulse interacts with, and emerges ahead of, the slower pulse, with the shape and the velocity of each soliton retained.

Remark.
The line-soliton solutions of the KP equation are an extension of those of the KdV equation to two space dimensions. When $k_y = 0$, Equation (25) represents a one-dimensional, one-soliton wave solution of the GKdV equation (2). To show the effectiveness of the proposed linearized implicit method (4) for the GKdV equation, two cases corresponding to $\alpha = 1$ and $\alpha = 2$ are described for the GKdV equation with soliton-type solutions (Fornberg and Whitham [17]). Here, the numerical boundary conditions can be taken to be zero for a large computational domain.

Figure 4 illustrates numerical results for the GKdV equation for the one soliton solution for $\beta = 6$, $\mu = 1$, $k_x = 0{\cdot}7$, $\Delta t = 0{\cdot}05$, $\Delta x = 0{\cdot}1$ and $\alpha = 1$ and $\alpha = 2$ using the implicit linearized method (4). From this figure, it is found that as α increases, the speed velocity of the wave decreases and the amplitude increases as time increases (that is, wave increases in

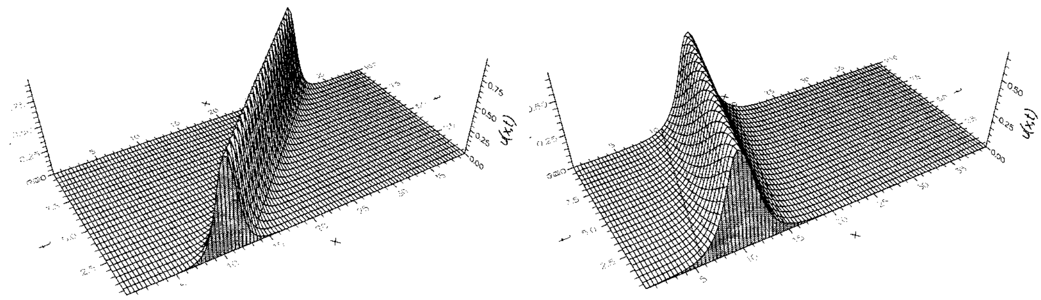

Figure 4. Solitary waves for the generalized KdV equation with $\alpha = 1$ and $\alpha = 2$ respectively.

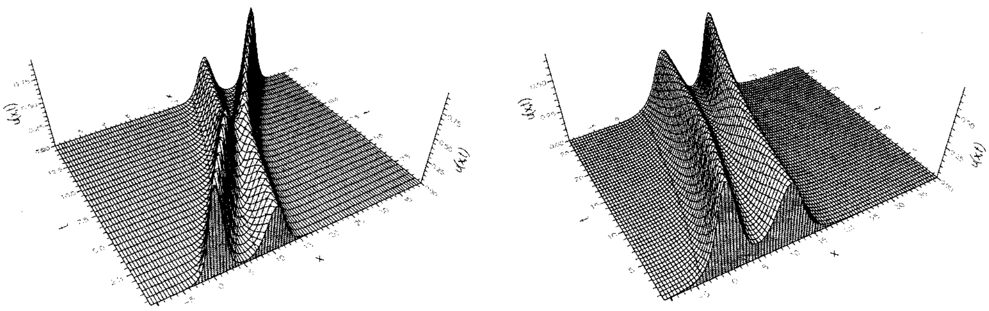

Figure 5. Solitary waves interactions for the generalized KdV equation with $\alpha = 1$ and $\alpha = 2$ respectively.

height and width). The numerical method was also run for other values of α and the results are found to be in good agreement with the corresponding exact solution $u(x, t)$ given by (25) with $k_y = 0$. The conserved quantities (the momentum $P(t) = \int_{-\infty}^{\infty} u \, \mathrm{d}x$ and the energy $E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u^2 \, \mathrm{d}x$) are used to provide a check on the numerical integrations. It is found that the quantities $P(t)$ and $E(t)$ remain constant with respect to the time t. Moreover, as α increases, the conserved quantities (momentum and the energy) increase.

Figure 5 shows the interaction of two soliton solutions of the GKdV equation for $\beta = 6$, $\mu = 1$, $k_{x_1} = 0{\cdot}7$, $k_{x_2} = 0{\cdot}5$, $x_1 = 0$, $x_2 = 5$, and $\alpha = 1$ and 2, respectively. As before, a numerical check on the conservation of momentum and energy shows that both quantities remain constant with respect to the time t.

The numerical results here are computed with initial conditions of the form

$$u(x, 0) = \sum_{i=1}^{2} A_i [\operatorname{sech}(k_{x_i} x - x_i)]^{2/\alpha}, \qquad -10 < x < 40, \tag{27}$$

where

$$A_i = \frac{2(\alpha + 1)(\alpha + 2)}{\beta\alpha^2}\mu k_{x_i}^2, \qquad \omega_i = \frac{4\mu k_{x_i}^3}{\alpha^2}$$

and x_1 and x_2 are the initial positions.

KP one and two lump soliton solutions:

The propagation of a two-dimensional solitary wave solution (and the interaction of two lump solitons) which decays to zero in all space directions are presented. For large enough compu-

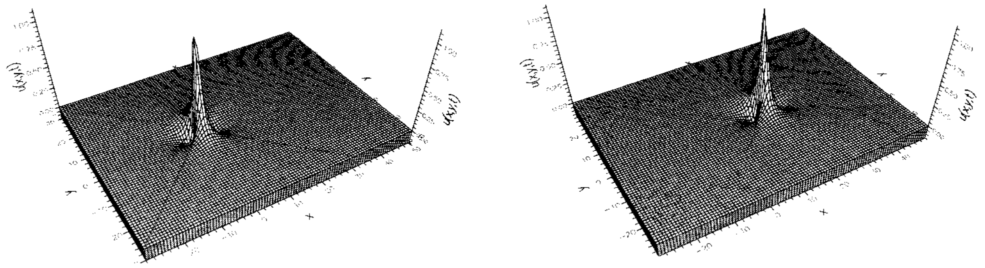

Figure 6. The initial condition (29) and the lump-like soliton at time $t = 12$ for the KP equation.

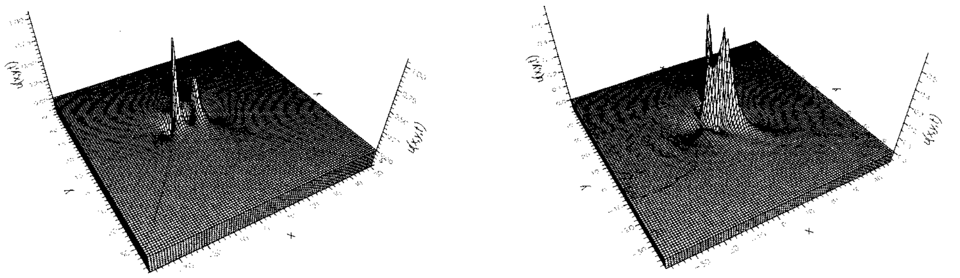

Figure 7. The initial condition (30) and the two lump soliton interaction at time $t = 8$ for the KP equation.

tational domain, the numerical boundary conditions along the y-axis can be taken to be the same as in (26), and along the x-axis they can be taken as

$$U^n_{m,k} = U^{n+1}_{m,k} = 0 \quad \text{for } k = 0, 1 \quad \text{and} \quad k = L, L+1 \tag{28}$$

and $m = 1, 2, \ldots, N$.

Numerical results for the one lump soliton obtained using the linearized implicit scheme (13) with $\beta = 6$, $\mu = 1$, $\delta = 1$, $\Delta t = 0{\cdot}05$, $\Delta x = 0{\cdot}2$, $\Delta y = 0{\cdot}5$ and tolerance $\varepsilon = 0{\cdot}0001$ are shown in Figure 6 at time $t = 12$, together with the initial condition (Minzoni and Smyth [22]) given by

$$u(x, y, 0) = \frac{4[-(x - x_0)^2 + w(y - y_0)^2 + 3/(\delta w)]}{[(x - x_0)^2 + w(y - y_0)^2 + 3/(\delta w)]^2}, \tag{29}$$

with $x_0 = y_0 = 0$ and $w = 1$.

The results in Figure 6 show that the solitary wave propagates stably in the x-direction.

Figures 7–8 show the interaction of two lump solitons at times $t = 8$ and $t = 18$, together with the initial condition given by

$$u(x, y, 0) = \sum_{i=1}^{2} \frac{4[-(x - x_i)^2 + w_i(y - y_i)^2 + 3/(\delta w_i)]}{[(x - x_i)^2 + w_i(y - y_i)^2 + 3/(\delta w_i)]^2}, \tag{30}$$

with $x_1 = -4$, $x_2 = 4$, $y_1 = y_2 = 10$, $w_1 = 1$ and $w_2 = 0{\cdot}5$.

From these figures, it can be seen that the faster pulse interacts with the slower pulse; the interaction is non-linear. It is also seen from Figure 8 that, after interaction, their amplitudes

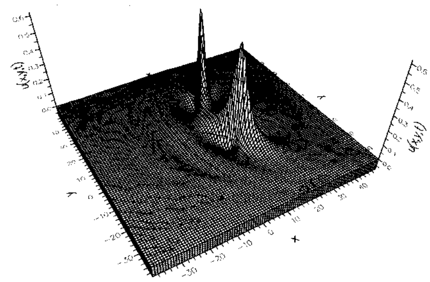

Figure 8. Two lump soliton interaction at time $t = 18$ for the KP equation.

become almost identical, and they propagate along the y-axis. A numerical check on the conserved quantities

$$P(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \, \mathrm{d}x \, \mathrm{d}y$$

and the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 \, \mathrm{d}x \, \mathrm{d}y$$

show that these quantities remain constant with respect to the time t.

Finally, the predictor-corrector method (14–15) was run for the one line-soliton of the KP equation, with $\beta = 6$, $\mu = 0{\cdot}01$, $\delta = 1$, $k_x = 0{\cdot}7$, $k_y = -0{\cdot}3$, $\Delta t = 0{\cdot}001$, $\Delta x = 0{\cdot}5$, $\Delta y = 0.5$ and $\eta = 7/6$ at time $t = 3$. The results obtained are found to be in good agreement with the analytical solution. Increasing the value of μ it is found that a smaller time step is needed in order that the predictor-corrector method (14–15) remains stable, and this would require a large number of time steps which, consequently, increases significantly the time computation. Interpreting ν in (16) to be the maximum value of u at the time level t_0, the chosen values of Δt, Δx and Δy do satisfy the stability restrictions (22), (24). Violation of these conditions leads to solutions that remain bounded for a period of time before experiencing a sudden growth leading to overflow within a few time steps.

4. Conclusions

Computational methods based on a linearized implicit scheme and a predictor–corrector method have been presented for the numerical solution of the KP equation and its generalized form. The methods developed were analysed using linear stability theory. It was found that the linearized implicit scheme is unconditionally stable, whereas the predictor–corrector is conditionally stable.

Numerical results for the KP (and GkdV) equations were reported for one line-soliton and the interaction of two line-solitons. A lump-like soliton and the interaction of two lump-like solitons were also reported for the KP equation. In the experiments for more than one soliton the usual features of retention of shape and velocity by each pulse were observed, indicating that the linearized implicit method presented, which was based on finite-difference replacements of derivatives, is appropriate for solving the KP equation. In future work, focus will be on the application of the linearized implicit method for the numerical simulation of the generalized KP equation.

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